Fluctuating hydrodynamics and turbulence in a rotating fluid: Universal properties

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Abstract

We analyze the statistical properties of three-dimensional ($3d$) turbulence in a rotating fluid. To this end we introduce a generating functional to study the statistical properties of the velocity field $\mathbf{v}$. We obtain the master equation from the Navier-Stokes equation in a rotating frame and thence a set of exact hierarchical equations for the velocity structure functions for arbitrary angular velocity $\mathbf{\Omega}$. In particular we obtain the *differential forms* for the analogs of the well-known von Karman-Howarth relation for $3d$ fluid turbulence. We examine their behavior in the limit of large rotation. Our results clearly suggest dissimilar statistical behavior and scaling along directions parallel and perpendicular to $\mathbf{\Omega}$. The hierarchical relations yield strong evidence that the nature of the flows for large rotation is not identical to pure two-dimensional flows. To complement these results, by using an effective model in the small-Ω limit, within a one-loop approximation, we show that the equal-time correlation of the velocity components parallel to $\mathbf{\Omega}$ displays Kolmogorov scaling $q^{-5/3}$, where as for all other components, the equal-time correlators scale as q^{-3} in the inertial range where $\mathbf{q}$ is a wavevector in $3d$. Our results are generally testable in experiments and/or direct numerical simulations of the Navier-Stokes equation in a rotating frame.

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I. INTRODUCTION

Near the second order phase transition equilibrium systems exhibit scaling behavior for thermodynamic functions and correlations. These are characterized by certain scaling exponents which depend on the spatial dimension d and the symmetry of the order parameter characterizing the phase transition. These, however, do not depend on the parameters specifying the Hamiltonian [1]. Time dependent correlation functions, characterized by dynamic scaling exponents also show similar universality [2]. These standard universal properties of equilibrium critical dynamics are fairly robust with respect to perturbations violating detailed balance [3]. In contrast, truly nonequilibrium systems, like fluid and magneto-hydrodynamic turbulence, surface growth etc., are described by appropriate equations of motion and exhibit much richer universal behavior. Non-equlibrium systems tend to be more sensitive on the parameters that appear in the equations of motion. For example, one finds that for the Kardar-Parisi-Zhang equation, anisotropic perturbations are relevant in $d > 2$ spatial dimensions, leading to rich phenomena that include novel universality classes and the possibility of first-order phase transitions and multicritical behavior [4].

Turbulence in fluid, described by the Navier-Stokes equation [5, 6] for the evolution of the velocity field $\mathbf{v}$, is a good candidate of systems out of equilibrium due to the external drive acting on the system. Statistically steady fluid turbulence in three- ($3d$) and two- ($2d$) dimensions show markedly different behavior: In $3d$, homogeneous and isotropic turbulence is characterized by a set of multiscaling exponents for the structure functions (see below) for distance r in the inertial range between the forcing scale L and the dissipation scale η_d (i.e., $\eta_d \ll r \ll L$), and forward cascade (from small to large wavenumbers) of the energy. Turbulence in $2d$ shows an inverse cascade of kinetic energy from the energy-injection scale to larger length scales and a direct cascade in which the enstrophy cascades towards small length scales [7]; in many physical realizations of $2d$ turbulence, there is an air-drag-induced friction. In this direct-cascade regime, velocity structure functions show simple scaling but their vorticity counterparts exhibit multiscaling [8], with exponents that depend on the friction. In a rotating fluid isotropic symmetry is broken by the global rotation. How the breakdown of rotational invariance affects the universal properties remains a very important theoretical question. These studies are also important for geophysical flows, e.g., flows in ocean and atmosphere. Ref. [9], in numerical simulations of forced rotating incompressible turbulence

within a periodic box of small aspect ratio, showed that above a critical rotation, $3d$ forcing leads to a $2d$ inverse cascade. Further Ref. [10], in a spectral approach to rotating turbulence applied to a specific eddy damped quasi-normal Markovian model, showed a trend towards two dimensional behavior in presence of rotation. Ref. [11] in a helical decomposition of the Navier-Stokes equation demonstrated similar trends of two dimensionalization of energy transfer. In recent studies using a shell model for rotating turbulent fluid, the authors showed how quasi two-dimensional behavior emerge as rotation speed Ω increases [12]. Recent Direct Numerical Study (DNS) [13] for rotating fluid turbulence suggest that the energy spectra for the velocity components perpendicular to $\mathbf{\Omega}$ scales as $q_\perp^{-2}$ in the inertial range, with wavevector $\mathbf{q}_\perp$ being perpendicular to $\mathbf{\Omega}$. It is not fully established whether the statistical nature of the flow for large-$\mathbf{\Omega}$ is truly two-dimensional or not. In this context, Ref. [14] in a helical decomposition argued that at the lowest order rotating turbulence is not the same as $2d$ turbulence.

Since a complete description of fully developed homogeneous and isotropic $3d$ turbulence requires enumeration of all the multiscaling exponents, it is important to find out the nature of multiscaling in the presence of rotation. Until now, there is no theoretical perturbative calculational framework to obtain these multiscaling exponents within controlled perturbative approximations. However, the multiscaling exponent for the third-order structure function for the longitudinal component of the velocity field is well-known: Defining $\Delta\mathbf{v}(x) = \mathbf{v}(\mathbf{x_1}+\mathbf{x}) - \mathbf{v}(\mathbf{x_1})$, one has in the inertial range

$$\langle[\Delta v(|\mathbf{r}|)\cdot\hat{\mathbf{r}}]^3\rangle = -\frac{4}{5}\epsilon r, \tag{1}$$

where ϵ is the total energy dissipation (see below) and $\hat{\mathbf{r}} = \mathbf{r}/r$. This is the well-known von Karman-Howarth 4/5-law of $3d$ fluid turbulence [15]. The corresponding differential form is given by

$$\nabla_j\langle\Delta v^2\Delta v_j\rangle = 4\epsilon. \tag{2}$$

Subsequently, an analog of Eq. (2), involving mixed correlation tensor of the velocity and vorticity, has been obtained for helical turbulence [16]. In particular they find correlation

$$\langle v_i(\mathbf{x})v_j(\mathbf{x})w_j(\mathbf{x}+\mathbf{r})\rangle = -\frac{\overline{\eta}}{10}r_i, \tag{3}$$

It remains to find out analogs of these relations as above in the presence of rotation, which introduces helicity in the system.

Our main results in this article are (obtained by using two different strategies):

- By using the Navier-Stokes equation in a rotating frame, we set up the master equation for the generating functional of the probability distribution of the velocity field differences. We use this to obtain the exact hierarchical relations between different order structure functions of velocity components. In particular we obtain a set of *differential forms* for the analog of the well-known von Karman-Howarth relation for non-rotating $3d$ fluid turbulence. These relations are, however, non-integrable and involve contributions from pressure, unlike non-rotating $3d$ turbulence. Thus we find that, unlike the case of fluid turbulence in an inertial frame, there are no closed equations for the third order structure functions and hence no simple analog of the well-known von Karman-Howarth relation for inertial frame fluid turbulence in rotating turbulence. A combination of these relations yield a compact differential form of von Karman-Howarth like relation, which although identical to its isotropic analog cannot be integrated owing to the underlying anisotropy. We examine the relations in the limit of large-Ω. In this limit, we obtain simple relations connecting velocity structure functions with mixed structure functions, constructed out of gradients of pressure differences and components of velocity differences. Finally, our results here are illustrative of the differences between the statistical properties of $3d$ turbulent flows in the presence of large rotation and those of pure $2d$ turbulence.

- From the Navier-Stokes equation in presence of rotation we define an *effective model* which is expected to be valid in the small-Ω limit. We apply mode-coupling methods on this effective model to calculate the equal-time two-point correlation functions of the velocity field components v_i, $i = x, y, z$. We find that for $i = z$, $\langle |v_z(\mathbf{q}, t)|^2| \rangle \sim q^{-11/3}$, where as $\langle v_i(\mathbf{q}, t) v_j(-\mathbf{q}, t) \rangle \sim q^{-3}$, $i, j \neq z$, where $\mathbf{\Omega}$ is along the z-direction and $\mathbf{q}$ is a $3d$ wavevector. Thus, correlations involving v_x or v_y exhibit scaling $\sim q^{-3}$ demonstrating the anisotropic nature of rotating turbulence.

The remaining part of the paper is organized as follows: In Sec. II we write down the Navier-Stokes equation in the presence of a global rotation. In Sec. III we set up the hierarchical relatione between equal-time structure functions of different order and the effects of rotation on them. We then define our *effective model* to study rotating turbulence in Sec. IV. We use it to perform a one-loop self-consistent (OLSC) calculation on the effective model to

calculate various equal-time velocity two-point correlation functions and enumerate their scaling in the inertial range. We examine the limit of large-$\mathbf{\Omega}$. Finally, we summarize and discuss our results in Sec. V.

II. EQUATION OF MOTION

The Navier-Stokes equation in presence of a global rotation $\mathbf{\Omega} = \Omega\hat{z}$ is given, in the rotating frame, by

$$\frac{\partial \mathbf{v}}{\partial t} + 2(\mathbf{\Omega} \times \mathbf{v}) + \mathbf{v}.\nabla\mathbf{v} = -\frac{\nabla p^*}{\rho} + \nu_0 \nabla^2 \mathbf{v} + \mathbf{f} \tag{4}$$

with $\nabla.\mathbf{v} = 0$ (conclusion of incompressibility). In Eq.(4), p^* is the effective pressure which includes the centrifugal force (through an effective potential $\frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^{\mathbf{2}}$), ν_0 is the kinematic viscosity and $\mathbf{f}$ is a large-scale external forcing function, required to maintain a statistical steady-state. The term $2(\mathbf{\Omega} \times \mathbf{v})$ is the Coriolis force. Note that Eq. (4) is invariant under

$$z \to -z,\, v_z \to -v_z. \tag{5}$$

However, the symmetry under $\mathbf{r}_\perp \to -\mathbf{r}_\perp$, $\mathbf{v}_\perp \to -\mathbf{v}_\perp$, which is present in the Navier-Stokes equation without rotation, is broken in Eq. (4) due to rotation, where $\mathbf{r}_\perp = (x, y)$, $\mathbf{v}_\perp = (v_x, v_y)$. Thus rotation breaks the invariance under in-plane parity inversion. In a nonequlibrium system there is no particular relation between the noise variance and the dissipation coefficient, unlike in equilibrium systems where such a relation exists due to the Fluctuation Dissipation Theorem [17]. A promising starting point for theoretical/analytical studies on homogeneous and isotropic fluid turbulence is the randomly forced Navier-Stokes model, where the forcing function $\mathbf{f}$ is a Gaussian random force whose spatial Fourier transform $\mathbf{f}(\mathbf{q}, t)$ has zero mean and covariance [18]

$$\langle f_i(\mathbf{q}, \omega) f_j(-\mathbf{q}, -\omega)\rangle = P_{ij}(\mathbf{q}) \frac{2D_0}{q^{d-4+y}} \tag{6}$$

in d-dimensions where $P_{ij} = \delta_{ij} - \frac{q_i q_j}{q^2}$ is the transverse projection operator in the Fourier space, and $\mathbf{q}$ and ω are wave vector and frequency respectively. This was used in Ref. [18] to calculate various universal quantities associated with $3d$ turbulence. The Model A (with $y = 2 - d$) and Model B (with $y = 4 - d$) of Ref. [19] may be considered as special cases of (6). In absence of any global rotation ($\mathbf{\Omega} = \mathbf{0}$) in $3d$, for $y = 4$ one obtains the famous

Kolmogorov (K41) result for the energy spectrum: $E(q) = K_o \epsilon^{2/3} q^{-5/3}$ for wavenumber k in the inertial range. Here K_o is the Kolmogorov's constant (a dimensionless universal number) and ϵ is energy dissipation rate per unit mass which is the flux of the energy in the steady state. In a rotating fluid system there are two important dimensionless numbers, namely, the Rossby number $Ro = V/(2\Omega L)$ and the Ekman number $M_0 = \nu_0/(2\Omega L^2)$ where V and L are the characteristic velocity and length scales, respectively. From Eq.(4) it is clear that the possible modifications of the statistical properties due to the rotation is essentially a non-linear effect, because without the nonlinearity, the rotation only causes the $x-$ and the $y-$components of $\mathbf{v}$ to rotate. From a statistical mechanics point of view, quantities of interests are the correlations $C_{ij}(q,\omega) = \langle v_i(\mathbf{q},\omega) v_j(-\mathbf{q},-\omega)\rangle$. For homogeneous and isotropic turbulence (i.e., for $\mathbf{\Omega} = 0$), $C_{ij}(q,\omega) \sim P_{ij}(\mathbf{q}) q^{-2\chi - z - 3} g(\omega/q^z)$ for wavevector $\mathbf{q}$ in the inertial range, χ and z are the roughness and the dynamic exponents respectively, g is a scaling function. Due to the Galilean invariance of Eq.(4) $\chi + z = 1$ exactly [18]. It remains to be seen how a finite rotation affects the scaling obtained in the isotropic situation.

III. HIERARCHICAL RELATIONS BETWEEN STRUCTURE FUNCTIONS

In this Section we set up the exact hierarchical equations between the two-point structure functions of various orders by generalizing the generating-functional method of Ref. [20]. These equations are however not closed. These exact relations, nevertheless, reveal explicitly the differences between the rotating and the non-rotating cases.

In order to find the hierarchy of equations for the equal-time structure functions in the statistical stationary state of the stochastically forced Navier-Stokes equation in a rotating frame for the velocity field v_i:

$$\begin{aligned} \mathcal{Z}(\lambda_1, \lambda_2, \mathbf{r}_1, \mathbf{r}_2, t) &\equiv \langle \exp(\lambda_1 \cdot \mathbf{v}(\mathbf{x}_1) + \lambda_2 \cdot \mathbf{v}(\mathbf{x}_2)\rangle = \langle \mathcal{Z}_0 \rangle \\ &= \int \mathcal{D}v_1 \mathcal{D}v_2 \mathcal{P}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}_1, \mathbf{x}_2, t) \exp[\lambda_1 \cdot \mathbf{v}(\mathbf{x}_1) + \lambda_2 \cdot \mathbf{v}(\mathbf{x}_2)], \end{aligned} \tag{7}$$

where $\mathcal{P}$ is the joint probability distribution of velocities $\mathbf{v}_1(\mathbf{x}_1,t), \mathbf{v}_2(\mathbf{x}_2,t)$. The Eq. of motion for $\mathcal{Z}$ may be derived in a straightforward way:

$$\frac{\partial \mathcal{Z}}{\partial t} = -\frac{\partial^2 \mathcal{Z}}{\partial \lambda_j^\mu \partial x_j^\mu} - 2\lambda_j^\mu \epsilon_{jps} \frac{\partial \mathcal{Z}}{\partial \lambda_s^\mu} \Omega_p + I_p + I_f + D, \tag{8}$$

where $D = \langle \nu_0 [\lambda_1 \cdot \nabla^2 \mathbf{v}(\mathbf{x}_1) + \lambda_2 \cdot \nabla^2 \mathbf{v}(\mathbf{x}_2)] \mathcal{Z}_0 \rangle$ are the anomaly terms, $I_p = \langle [\lambda_1 \cdot \nabla p^*(\mathbf{x}_1) + \lambda_2 \cdot \nabla p^*(\mathbf{x}_2)] \mathcal{Z}_0 \rangle$ are the pressure contributions, and $I_f = \langle [\lambda_1 \cdot \mathbf{f}(\mathbf{x}_1) + \lambda_2 \cdot \mathbf{f}(\mathbf{x}_2)] \mathcal{Z}_0 \rangle$. Note

that the master equation (8) is not closed because of the anomaly terms D. This nontrivial point has been treated by a variety of approaches ranging from approximate techniques to rigorous studies [20, 21]. The problem arises when we look at the master equation in the limit of vanishing viscosity $\nu_0 \to 0$; here the anomaly terms produce a finite effect. For instance, the finiteness of the dissipation in the limit of vanishing viscosity (discussed above) is just produced by the anomaly terms D. In what follows we shall specify the effects of the anomaly terms in a more detailed way. Further, we can apply the Furutsu-Novikov-Donsker [22, 23] formalism to calculate the random forcing terms. For Gaussian distributed, white-in-time random forces $\mathbf{f}$, which are additive terms in Eq. (4), we obtain the the final form of *the master equation* for equal-time, two-point generating functions $\mathcal{Z}$:

$$\frac{\partial \mathcal{Z}}{\partial t} = -\frac{\partial^2 \mathcal{Z}}{\partial \lambda_j^\mu \partial x_j^\mu} - 2\lambda_j^\mu \epsilon_{jps} \frac{\partial \mathcal{Z}}{\partial \lambda_s^\mu} \Omega_p + I_p + D + [K(0) - K(r)]\mathcal{Z}, \tag{9}$$

where $K(r)$ is given by $\langle \mathbf{v}(r,t) \cdot \mathbf{f}(0,t) \rangle$ (to be calculated by using the Furutsu-Novikov-Donsker formalism), which is related to the variance of $\mathbf{f}$.

In order to proceed further it is useful to apply the basic symmetries of the dynamical equations to simplify the structure of the master equation. We assume that statistically stationary turbulence has been produced under the dynamics of the stochastically forced Navier-Stokes Eq. (4). Stationarity implies

$$\partial_t \mathcal{Z} = 0. \tag{10}$$

The Navier-Stokes equation without rotation (i.e., Eq.(4) with $\mathbf{\Omega} = 0$) is invariant under uniform translation. For $\mathbf{\Omega} \neq 0$, the presence of the centrifugal force $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$, which is contained inside effective pressure p^*, breaks the translational invariance. Since we are considering incompressible flows, pressure p^* may be eliminated by using the condition $\nabla \cdot \mathbf{v} = 0$. We obtain

$$-\nabla^2 p^* = \nabla \cdot (\mathbf{\Omega} \times \mathbf{v}) + \nabla_j (\mathbf{v} \cdot \nabla) v_j. \tag{11}$$

The resulting equation after eliminating p^* then becomes

$$\frac{\partial v_i}{\partial t} + 2P_{ij}(\mathbf{\Omega} \times \mathbf{v})_j + P_{ij}\mathbf{v} \cdot \nabla v_j = \nu_0 \nabla^2 v_i + f_i. \tag{12}$$

Evidently, Eq. (12) is invariant under $\mathbf{r} \to \mathbf{r} + \mathbf{r}_0$ where $\mathbf{r}_0$ is a constant vector. Thus for the hierarchical relations between structure functions for an incompressible fluid, we impose

homogeneity on $\mathcal{Z}$:

$$\mathcal{Z} = \mathcal{Z}(\lambda_1, \lambda_2, \mathbf{x}_1 - \mathbf{x}_2). \tag{13}$$

Equivalently, if we define

$$\mathbf{x}^+ \equiv \mathbf{x}_1 + \mathbf{x}_2, \;\; \mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2, \tag{14}$$

we can write

$$\frac{\partial}{\partial x_{1i}}\mathcal{Z} = \frac{\partial}{\partial x_{+i}}\mathcal{Z} + \frac{\partial}{\partial r_i}\mathcal{Z}, \tag{15}$$

$$\frac{\partial}{\partial x_{2i}}\mathcal{Z} = \frac{\partial}{\partial x_{i+}}\mathcal{Z} - \frac{\partial}{\partial x_{ri}}\mathcal{Z}. \tag{16}$$

Homogeneity now implies $(\partial/\partial x_{i+})\mathcal{Z} = 0$.

The $3d$ Navier Stokes Equation (4) is invariant under the Galilean transformation

$$\mathbf{x} = \mathbf{x}' + \mathbf{u}_0 t', \;\; t = t', \;\; \mathbf{v}(\mathbf{x}, t) = \mathbf{v}'(\mathbf{x}', t') + \mathbf{u}_0. \tag{17}$$

Since $\mathcal{Z} = \langle \exp(\lambda_1 \cdot \mathbf{v}(\mathbf{x}_1) + \lambda_2 \cdot \mathbf{v}(\mathbf{x}_2) \rangle$, this Galilean invariance implies

$$\mathcal{Z} = \mathcal{Z}(\lambda_1 - \lambda_2, \mathbf{x}_1 - \mathbf{x}_2). \tag{18}$$

If we now introduce variables

$$\begin{aligned} \lambda_0 &= \lambda_1 + \lambda_2, \\ \lambda &= \lambda_1 - \lambda_2, \end{aligned} \tag{19}$$

and use the same considerations as in Eqs. (15) and (16) for $\mathbf{x}_1$, $\mathbf{x}_2$, we see that the Galilean invariance is equivalent to demanding $\frac{\partial}{\partial \lambda_{1i}}\mathcal{Z} = 0 = \frac{\partial}{\partial \lambda_{2i}}\mathcal{Z}$. Therefore the master equation can be conveniently rewritten in terms of the variables λ and $\mathbf{r}$ to get the generating function

$$\mathcal{Z}(\mathbf{r}, \lambda) \;=\; \langle \exp[\lambda \cdot \mathbf{\Delta v}] \rangle \equiv \langle \mathcal{Z}_0 \rangle, \;\; (\text{with } \mathbf{\Delta v} = \mathbf{v}(\mathbf{x}_1) - \mathbf{v}(\mathbf{x}_2)\,), \tag{20}$$

which obeys the following equation in steady-state:

$$\frac{\partial^2 \mathcal{Z}}{\partial \lambda_j \partial r_j} + 4\lambda_j \epsilon_{jzs} \frac{\partial \mathcal{Z}}{\partial \lambda_s}\Omega = I_p + I_f + D, \tag{21}$$

where we have used $\mathbf{\Omega} = \Omega\hat{z}$. Due to the rotation (about the z-axis), the system is anisotropic and $3d$ rotational invariance is broken. However the system admits *restricted*

two-dimensional rotational invariance about the z-axis. This dictates that $\mathcal{Z}$ depends separately on z and $\mathbf{r}_\perp$ ($\mathbf{r}_\perp$ lies in the xy-plane). We define $\lambda = (\lambda_z,\, \eta_1,\, \eta_2)$, i.e., $\lambda_z,\, \eta_1,\, \eta_2$ are the z, r and θ-components of λ, respectively. In terms of these variables $\mathcal{Z}$ may be expressed as

$$\mathcal{Z} = \langle \exp[\lambda_z \Delta v_z(z, \mathbf{r}_\perp) + \eta_1 \Delta v_r(z, \mathbf{r}_\perp) + \eta_2 \Delta v_\theta(z, \mathbf{r}_\perp)]\rangle, \tag{22}$$

where $\Delta v_z, \Delta v_r$ and Δv_θ are z, r and θ components of the velocity difference vector $\mathbf{\Delta v}$. The anomaly terms D may be written in terms of the energy dissipation rate per unit mass ϵ as given below (see Appendix for details)

$$\begin{aligned} D \;&=\; -2\nu_0 \langle [\lambda_z^2 (\nabla_i v_z)^2 + \eta_1^2 (\nabla_i v_r)^2 + \eta_2^2 (\nabla_i v_\theta)^2 + 2\lambda_z \eta_1 (\nabla_i v_z)(\nabla_i v_r) \\ &+\; 2\eta_1 \eta_2 (\nabla_i v_r)(\nabla_i v_\theta) + 2\eta_2 \lambda_z (\nabla_i v_\theta)(\nabla_i v_z)] \mathcal{Z}_0 \rangle. \end{aligned} \tag{23}$$

In a statistical steady state, these terms are nonzero constants and are related to the energy input that is determined by the variances of the external stochastic forces. We are interested in the scaling properties of various structure functions in the inertial range ($r \ll L = 1$, where L is the system size), where the force correlator $K(r)$ may be expanded in powers of r. To the leading order we assume the following Taylor expansion: $K(r) = K(0) - Ar^2$, where A is a numerical constant. We then neglect the r^2 term relative to contributions from D. We now proceed to calculate hierarchical relations between structure functions of different orders. It is seen easily that differentiation with respect to $\lambda_z,\, \eta_1,\, \eta_2$ leads to various structure functions of different order: We write $S_{m,n,p} = \langle (\Delta v_z)^m (\Delta v_r)^n (\Delta v_\theta)^p \rangle = \frac{\partial^{m+n+p}}{\partial \lambda_z^m \partial \eta_1^n \partial \eta_2^p} \mathcal{Z}|_{\lambda_z=0, \eta_1=0, \eta_2=0}$. Now $\mathcal{Z}$ as a function of $\lambda_z,\, \eta_1,\, \eta_2$ as defined in (22) satisfies the steady-state equation

$$\begin{aligned} &\frac{\partial^2 \mathcal{Z}}{\partial z \partial \lambda_z} + \frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}\left(r_\perp \frac{\partial \mathcal{Z}}{\partial \eta_1}\right) + \frac{1}{r_\perp}\left(-\eta_2 \frac{\partial}{\partial \eta_1} + \eta_1 \frac{\partial}{\partial \eta_2}\right)\frac{\partial \mathcal{Z}}{\partial \eta_2} + 4\Omega\left(-\eta_1 \frac{\partial}{\partial \eta_2} + \eta_2 \frac{\partial}{\partial \eta_1}\right)\mathcal{Z} \\ &= \; I_p + I_f + D. \end{aligned} \tag{24}$$

It will now be useful to consider the pressure contribution: The Navier-Stokes Equation in an inertial frame ($\Omega = 0$) is invariant under the parity transformation: $\mathbf{r} \to -\mathbf{r}$, $\mathbf{v} \to -\mathbf{v}$, where $\mathbf{r}$ is the $3d$ radius vector. This yields that there are no pressure contributions to the well-known von Karman-Howarth relation for the third order longitudinal velocity structure function in non-rotating turbulence [6, 24, 25]. When there is a finite rotation ($\Omega \neq 0$), there is no invariance under the parity transformation above, and as a result, certain pressure contributions are finite (see below) which are zero in the non-rotating case. In addition, the

effective pressure p^* now explicitly depends on Ω as given by Eq. (11), and hence will be large in the large-Ω limit. With this discussion in mind, let us now write down the hierarchical relations between structure functions of various orders: Separately apply $\frac{\partial}{\partial\lambda_z}$, $\frac{\partial}{\partial\eta_1}$, $\frac{\partial}{\partial\eta_2}$ and setting λ_z, η_1, η_2 to zero yield

$$\begin{aligned} \frac{\partial}{\partial z}\langle(\Delta v_z)^2\rangle + \frac{\partial}{\partial r_\perp}\langle\Delta v_r\Delta v_z\rangle + \frac{1}{r_\perp}\langle\Delta v_r\Delta v_z\rangle &= 0, \\ \frac{\partial}{\partial z}\langle\Delta v_z\Delta v_r\rangle + \frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}\langle r_\perp(\Delta v_r)^2\rangle + \frac{1}{r_\perp}\langle(\Delta v_\theta)^2\rangle &= 0, \\ \frac{\partial}{\partial z}\langle\Delta v_z\Delta v_\theta\rangle + \frac{\partial}{\partial r_\perp}\langle\Delta v_r\Delta v_\theta\rangle &= 0. \end{aligned} \tag{25}$$

Note the above relations between the second order structure functions are independent of $\mathbf{\Omega}$. In order to calculate similar relations between different third order structure functions, we separately apply $\frac{\partial}{\partial\lambda_z^2}$, $\frac{\partial}{\partial\eta_1^2}$, and $\frac{\partial}{\partial\eta_2^2}$ and set λ_z, η_1, η_2 to zero. We obtain

$$\begin{aligned} &\frac{\partial}{\partial z}\langle(\Delta v_z)^3\rangle + \frac{\partial}{\partial r_\perp}\langle(\Delta v_z)^2\Delta v_r\rangle + \frac{1}{r_\perp}\langle(\Delta v_z)^2\Delta v_r\rangle = -4\langle\epsilon_z\rangle, \\ &\frac{\partial}{\partial z}\langle\Delta v_z(\Delta v_r)^2\rangle + \frac{\partial}{\partial r_\perp}\langle(\Delta v_r)^3\rangle + \frac{1}{r_\perp}\langle(\Delta v_\theta)^2\Delta v_r\rangle + \frac{1}{r_\perp}\langle(\Delta v_r)^3\rangle - 4\Omega\langle\Delta v_\theta\Delta v_r\rangle \\ =\ & -2\langle[\frac{\partial\Delta p^*}{\partial r_\perp}\Delta v_r]\rangle - 4\langle\epsilon_r\rangle, \\ &\frac{\partial}{\partial z}\langle\Delta v_z(\Delta v_\theta)^2\rangle + \frac{\partial}{\partial r_\perp}\langle\Delta v_r(\Delta v_\theta)^2\rangle + 4\Omega\langle\Delta v_\theta\Delta v_r\rangle \\ =\ & -4\langle\epsilon_\theta\rangle. \end{aligned} \tag{26}$$

Here $\epsilon_z = \nu_0(\nabla_i v_z)^2$, $\epsilon_r = \nu_0(\nabla_i v_r)^2$, $\epsilon_\theta = \nu_0(\nabla_i v_\theta)^2$, pressure difference $\Delta p^* = p^*(\mathbf{x}_1) - p^*(\mathbf{x}_2)$. Adding the three relations (26) we obtain

$$\begin{aligned} &\frac{\partial}{\partial z}\langle(\Delta v_z)^3\rangle + \frac{\partial}{\partial r_\perp}\langle(\Delta v_z)^2\Delta v_r\rangle + \frac{1}{r_\perp}\langle(\Delta v_z)^2\Delta v_r\rangle \\ +\ & \frac{\partial}{\partial z}\langle\Delta v_z(\Delta v_r)^2\rangle + \frac{\partial}{\partial r}\langle(\Delta v_r)^3\rangle + \frac{1}{r_\perp}\langle(\Delta v_r)^3\rangle \\ +\ & \frac{\partial}{\partial z}\langle\Delta v_z(\Delta v_\theta)^2\rangle + \frac{\partial}{\partial r_\perp}\langle\Delta v_r(\Delta v_\theta)^2\rangle + \frac{1}{r_\perp}\langle(\Delta v_\theta)^2\Delta v_r\rangle \\ =\ & -2\langle[\frac{\partial\Delta p^*}{\partial r_\perp}\Delta v_r]\rangle - 4\langle\epsilon\rangle. \end{aligned} \tag{27}$$

Here $\langle\epsilon\rangle$ is the total energy dissipation rate per unit mass: $\langle\epsilon\rangle = \langle\epsilon_z\rangle + \langle\epsilon_r\rangle + \langle\epsilon_\theta\rangle$. Relation (27) may be written in a closed form, by using incompressibility of the fluid, as

$$\nabla_j\left[\langle|\Delta\mathbf{v}|^2\Delta v_j\rangle + \langle\Delta p^*\Delta v_j\rangle\right] = -4\langle\epsilon\rangle, \tag{28}$$

where we have used that $\langle \frac{\partial \Delta p^*}{\partial z} \Delta v_z \rangle = 0$ due to the invariance of the Navier-Stokes equation (4) under $z \to -z$, $v_z \to -v_z$. Further, $|\Delta \mathbf{v}|^2 = (\Delta v_z)^2 + (\Delta v_r)^2 + (\Delta v_\theta)^2$. Equation (28) is the *differential* analog of the well-known von Karman-Howarth relation of fluid turbulence without rotation [6]. This may be simplified further by using incompressibility: One obtains $\nabla_j \langle |\Delta \mathbf{v}|^2 \Delta v_j \rangle = 0$, which is same as in Ref. [26]. Although it is identical to the von Karman-Howarth relation (2), it cannot be integrated due to the anisotropy. Thus, in that sense, a simple (integral) analog of the von Karman-Howarth relation for rotating fluid turbulence does not exist. Equation (28) is one of the main results of this article. When $\mathbf{\Omega} = 0$, the pressure contributions in Eq. (26) vanish and consequently one obtains the result for the nonrotating case. Three more relations, analogous to (26) may be obtained by separately applying $\frac{\partial}{\partial \lambda_z} \frac{\partial}{\partial \eta_1}$, $\frac{\partial^2}{\partial \lambda_z \partial \eta_2}$, $\frac{\partial^2}{\partial \eta_1 \partial \eta_2}$ on Eq. (24) and setting λ_z, η_1, η_2 to zero. We find

$$
\begin{aligned}
&\frac{\partial}{\partial z} \langle (\Delta v_z)^2 \Delta v_r \rangle + \frac{\partial}{\partial r_\perp} \langle \Delta v_z (\Delta v_r)^2 \rangle + \frac{1}{r_\perp} \langle \Delta v_z (\Delta v_r)^2 \rangle + \frac{1}{r_\perp} \langle \Delta v_z (\Delta v_\theta)^2 \rangle - 4\Omega \langle \Delta v_z \Delta v_\theta \rangle \\
=\;& -\langle [\frac{\partial \Delta p^*}{\partial z} \Delta v_r] \rangle - \langle [\frac{\partial \Delta p^*}{\partial r_\perp} \Delta v_z] \rangle, \\
&\frac{\partial}{\partial z} \langle (\Delta v_z)^2 \Delta v_\theta \rangle + \frac{\partial}{\partial r_\perp} \langle \Delta v_z \Delta v_r \Delta v_\theta \rangle + 4\Omega \langle \Delta v_z \Delta v_r \rangle \\
=\;& -\langle [\frac{\partial \Delta p^*}{\partial z} \Delta v_\theta] \rangle, \\
&\frac{\partial}{\partial z} \langle \Delta v_z \Delta v_r \Delta v_\theta \rangle + \frac{\partial}{\partial r_\perp} \langle (\Delta v_r)^2 \Delta v_\theta \rangle + \frac{1}{r_\perp} \langle (\Delta v_\theta)^2 \Delta v_r \rangle + 4\Omega [-\langle (\Delta v_\theta)^2 \rangle + \langle (\Delta v_r)^2 \rangle] \\
=\;& -4\langle \epsilon_{r\theta} \rangle - \langle [\frac{\partial \Delta p^*}{\partial r} \Delta v_\theta] \rangle,
\end{aligned}
\tag{29}
$$

where $\epsilon_{r\theta} = \nu_0 (\nabla_j v_r)(\nabla_j v_\theta)$ is non-zero in the presence of rotation, but zero when $\mathbf{\Omega} = 0$ (due to the invariance under the in-plane parity inversion). Similar to Eqs. (26), Eqs. (29) do not decouple, have second order structure functions and pressure contributions. In particular, with the exception of the two-point structure function (in our notation defined above) $S_{2,0,0} \equiv \langle (\Delta v_z)^2 \rangle$, all other two-point velocity structure functions appear in the relations involving third-order structure functions (27) and (29). Despite the limitations of these relations, useful information may be obtained in the limit $\Omega \to \infty$ from relations (26) and (29). Note

that, from Eq. (11), in the limit p^*/Ω does not vanish, and hence we obtain

$$\begin{aligned}
\langle \Delta v_\theta \Delta v_r \rangle &= \frac{1}{2\Omega} \langle (\frac{\partial}{\partial r} \Delta p^*) \Delta v_r \rangle = 0, \\
\langle \Delta v_z \Delta v_\theta \rangle &= \frac{1}{4\Omega} [\langle \frac{\partial \Delta p^*}{\partial z} \Delta v_r \rangle + \langle \frac{\partial \Delta p^*}{\partial r} \Delta v_z \rangle], \\
\langle \Delta v_z \Delta v_r \rangle &= -\frac{1}{4\Omega} \langle \frac{\partial \Delta p^*}{\partial z} \Delta v_\theta \rangle, \\
-\langle (\Delta v_\theta)^2 \rangle + \langle (\Delta v_r)^2 \rangle &= -\frac{1}{4\Omega} \langle \frac{\partial \Delta p^*}{\partial r_\perp} \Delta v_\theta \rangle.
\end{aligned} \tag{30}$$

These are *exact* relations between certain two-point velocity structure functions and two-point mixed structure function of pressure gradients and velocity differences in the limit of large rotation. Thus we find that in this limit (i) $\langle \Delta v_\theta \Delta v_r \rangle = 0$, (ii) second order structure functions $\langle \Delta v_z \Delta v_\theta \rangle$, $\langle \Delta v_z \Delta v_r \rangle$ and the combination $\langle (\Delta v_r)^2 \rangle - \langle (\Delta v_\theta)^2 \rangle$ are simply related to certain mixed structure functions involving pressure difference gradient and velocity difference, (iii) second order structure functions $S_{2,0,0} = \langle (\Delta v_z)^2 \rangle$ has no simple relation with any mixed structure function. This is a possible indication of $S_{2,0,0}$ having different scaling properties in the large-$\mathbf{\Omega}$ limit, a fact we demonstrate below explicitly in a perturbative calculation by using an *effective model*. Finally, one may eliminate pressure p^* from the exact hierarchical relations (28) and (29) by using Eq. (11), although the ensuing forms of Eqs. (26) and (29) are not particularly illuminating (we do not report their explicit forms here). Let us now consider our results as above in the context of the helical analog of the von Karman-Howarth relation as studied in Ref. [16]. It may be noted that Ref. [16] considers helical turbulence which is still isotropic, where as rotating turbulence as considered here not only has finite helicity injected into the system by the rotation, the system no longer remains isotropic due to the existence of a preferred direction (the rotation axis). As a result, Ref. [16] has been able to obtain a closed form explicit analog of the von Karman-Howarth relation (1) for isotropic non-helical $3d$ turbulence. In contrast, we are able to obtain only a differential form of a von Karman-Howarth like relation for rotating turbulence. Apart from the anisotropy effects, yet another notable difference between our results and those in Ref. [16] is the nonlocal nature of the relations for rotating turbulence. Thus, despite the presence of helicity in both the cases, our results are not exactly equivalent of those in Ref. [16] in any special limit.

In the same way as above, analogous *exact* relations between certain third order velocity structure functions and mixed third order structure functions involving two factors of velocity

differences and one factor of pressure gradient difference are obtained in the large rotation limit. We write some of them below:

$$
\begin{aligned}
-2\langle(\Delta v_\theta)^2\Delta v_r\rangle + \langle(\Delta v_r)^3\rangle &= -\frac{1}{2\Omega}\langle(\frac{\partial}{\partial r_\perp}\Delta p^*)\Delta v_\theta\Delta v_r\rangle, \\
-\langle(\Delta v_\theta)^3\rangle + 2\langle(\Delta v_r)^2\Delta v_\theta\rangle &= -\frac{1}{4\Omega}\langle(\frac{\partial}{\partial r_\perp}\Delta p^*)(\Delta v_\theta)^2\rangle, \\
-2\langle(\Delta v_\theta)^2\Delta v_z\rangle + 2\langle(\Delta v_r)^2\Delta v_z\rangle &= -\frac{1}{4\Omega}[\langle\frac{\partial\Delta p^*}{\partial z}\Delta v_r\Delta v_\theta\rangle + \langle\frac{\partial\Delta p^*}{\partial r_\perp}\Delta v_\theta\Delta v_z\rangle], \\
-2\langle\Delta v_r\Delta v_\theta\Delta v_z\rangle &= \frac{1}{4\Omega}[\langle\frac{\partial\Delta p^*}{\partial r_\perp}(\Delta v_\theta)^2\rangle + 2\langle\frac{\partial\Delta p^*}{\partial r_\perp}\Delta v_z\Delta v_r\rangle]. \qquad (31)
\end{aligned}
$$

Relations (31) may further be rewritten in terms of $\mathbf{v}$ only by eliminating pressure using Eq. (11). Again, as for the second order functions, third order function $\langle(\Delta v_z)^3\rangle$ does not appear in these relations. This again is an indirect demonstration of anisotropic effect of rotation. Relations (25) and (30) together constitute a set of exact relations between different second order structure functions in the limit of $\Omega \to \infty$. Similarly, relations (26), (29) and (31) together constitute a complete set (differential) relations involving different third order structure functions for rotating turbulence in the large-$\mathbf{\Omega}$ limit. One may further obtain for any odd positive integer n

$$\langle\frac{\partial\Delta p^*}{\partial z}(\Delta v_z)^n\rangle = 0 \qquad (32)$$

for large rotation. Although, these hierarchical relations do not yield the scaling exponents directly, they serve as benchmarks on any theoretical (analytical or numerical) and experimental results on the structure functions. In order to progress further from Eqs. (30) or (31), one must make further approximations, see, e.g., Ref. [27]. Lastly, how do the relations (30) and (31) compare with those for pure $2d$ turbulence? As shown in Ref. [24], for pure $2d$ turbulence, pressure does not contribute to the relations between second order structure functions or third order structure functions. More generally, the large-Ω limit of the steady-state master equation (24) is not the same as that for $d = 2$: The master equation in the large-$\mathbf{\Omega}$ limit is

$$4\left(-\eta_1\frac{\partial}{\partial\eta_2} + \eta_2\frac{\partial}{\partial\eta_1}\right)\mathcal{Z} = I_p/\Omega, \qquad (33)$$

as all other terms in Eq. (24) vanish in the large-$\mathbf{\Omega}$ limit. The corresponding master equation for pure $2d$ turbulence in the steady state is given by [24]

$$\left[\frac{\partial}{\partial r_\perp}\frac{\partial}{\partial\eta_2} + \frac{2}{r_\perp}\frac{\partial}{\partial\eta_2} + \frac{\eta_3}{r_\perp}\frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_3} - \frac{\eta_2}{r_\perp}\frac{\partial^2}{\partial\eta_3^2}\right]\mathcal{Z}_2 = I_{f2} + I_{p2} + D_2, \qquad (34)$$

where $\mathcal{Z}_2$ is the $2d$ analog of Eq. (22), I_{f2}, I_{p2} and D_2 are the force contributions, pressure contributions and anomaly terms in $2d$. Clearly, Eq. (24), the master eqution for $3d$ rotating turbulence, is different from Eq. (34), the master equation for $2d$ non-rotating turbulence. This led us to generally conclude that the statistical properties of a turbulent rotating fluid in the large rotation limit is different from pure $2d$ turbulence.

Having discussed anisotropic effects of rotation on the statistical properties of $3d$ homogeneous turbulence in terms of the hierarchical relations between structure functions of different orders, we now set out to calculate the scaling of the equal time two-point velocity correlation functions (equivalently second order structure functions) explicitly within a one-loop perturbation calculation.

IV. EFFECTS OF ROTATION: PERTURBATIVE ANALYSIS

In this Section we use the Navier-Stokes equation (4) in a rotating frame, subject to stochastic forcings of the type (6). This has a long history in homogeneous and isotropic fluid turbulence studies which are well-documented in Refs. [28]; see also Ref. [29] for discussions on anisotropy and helicity using randomly stirred Navier-Stokes equation. Despite some well-known limitations and difficulties of the method it has succeeded in predicting several universal numbers and scaling exponents which are close to their experimentally obtained values. Note in the presence of rotation the Coriolis force introduces anisotropy even at the linear level in the Navier-Stokes equation [Eq.(4)]. Increasing rotation velocity $\mathbf{\Omega}$ (i.e., decreasing Ekman number M) is expected to modify the statistical behavior. In the Fourier space the Navier-Stokes Eq. (4) takes the form

$$\begin{aligned}&(-i\omega + \nu_0 k^2)v_i + i\frac{\lambda}{2}P_{ijp}(\mathbf{k})\Sigma_{\mathbf{q}}v_j(\mathbf{q},\omega_1)v_p(\mathbf{k}-\mathbf{q},\omega-\omega_1) - ik_i p^* \\ &= f_i - 2P_{im}(\mathbf{k})\epsilon_{mjp}\Omega_j v_p(\mathbf{k},\omega),\end{aligned} \tag{35}$$

where $\mathbf{k}$ is a $3d$ wavevector. Here, density ρ has been set to unity. The inverse of the bare propagator matrix G_0^{-1} is given by

$$G_0^{-1} = \begin{bmatrix} -i\omega + \nu_0 k^2 + 2P_{xy}\epsilon_{yzx}\Omega & 2P_{xx}\epsilon_{xzy}\Omega & 0 \\ 2P_{yy}\epsilon_{yzx}\Omega & -i\omega + \nu_0 k^2 + 2P_{yx}\epsilon_{xzy}\Omega & 0 \\ 2P_{zy}\epsilon_{yzx}\Omega & 2P_{zx}\epsilon_{xzy}\Omega & -i\omega + \nu_0 k^2 \end{bmatrix}. \tag{36}$$

Here $\epsilon_{ijk},\ i,j,k = x,y$ or z, is the totally antisymmetric tensor in $3d$: $\epsilon_{xyz} = 1$ etc. Different elements of G_0 are given in Sec. VIII. The renormalized correlation matrix $C_{ij}(\mathbf{k},\omega)$ is formally given by

$$C_{ij}(\mathbf{k},\omega) \equiv \langle v_i(\mathbf{k},\omega)v_j(-\mathbf{k},-\omega\rangle = G_{im}(\mathbf{k},\omega)\langle f_m(\mathbf{k},\omega)f_p(-\mathbf{k},-\omega)\rangle G_{pj}(-\mathbf{k},-\omega), \tag{37}$$

where $G_{ij}(\mathbf{k},\omega)$ is the fully renormalized version of the bare propagator $G_{ij0}(\mathbf{k},\omega)$ given by the matrix (36).

To find out non-linear effects of rotation and the renormalized correlation function, we now need to find out the fluctuation corrections to the different elements of G_0 in a systematic perturbation theory. It is evident that any straight forward perturbation theory will be much more algebraically complicated due to the non-isotropic nature of $G_0(\mathbf{k},\omega)$, reflected by the last term in the right hand side of Eq. (35), than its isotropic counter part [18]. In order to circumvent this algebraic difficulty we use a *modified* equation as an *effective* equation to calculate the renormalization of the elements of G_0. We replace, in the last term of Eq. 35, $v_p(\mathbf{k},\omega)$ by $\hat{G}_0(\mathbf{k},\omega)f_p(\mathbf{k},\omega)$: Thus we obtain

$$\begin{aligned}&\hat{G}_0^{-1}v_i + i\frac{\lambda}{2}P_{ijp}(\mathbf{k})\Sigma_{\mathbf{q}}v_j(\mathbf{q},\omega_1)v_p(\mathbf{k}-\mathbf{q},\omega-\omega_1) - ik_ip^* \\ &= f_i - 2\epsilon_{ijp}\Omega_j\hat{G}_0(\mathbf{k},\omega)f_p(\mathbf{k},\omega).\end{aligned} \tag{38}$$

Here, $\hat{G}_0$ is the bare propagator of Eq. (38):

$$\hat{G}_0(\mathbf{k},\omega) = \frac{1}{-i\omega + \nu_0k^2}. \tag{39}$$

Further the effective pressure may be eliminated by using the incompressibility constraint $\nabla\cdot\mathbf{v} = 0$. We obtain

$$\begin{aligned}&(-i\omega+\nu_0k^2)v_i + i\frac{\lambda}{2}P_{ijp}(\mathbf{k})\Sigma_{\mathbf{q}}v_j(\mathbf{q},\omega_1)v_p(\mathbf{k}-\mathbf{q},\omega-\omega_1) \\ &= f_i - 2P_{im}(\mathbf{k})\epsilon_{mjp}\Omega_j\hat{G}_0(\mathbf{k},\omega)f_p(\mathbf{k},\omega) \\ &\equiv \phi_i(\mathbf{k},\omega).\end{aligned} \tag{40}$$

The correlation of the effective noise ϕ_i is zero-mean, Gaussian distributed with a variance

$$\begin{aligned}&\langle\phi_i(\mathbf{k},\omega)\phi_j(-\mathbf{k},-\omega)\rangle \\ &= 2D_0|k^{-y}|P_{ij} - 4P_{jm}\epsilon_{mnp}\hat{G}_0(-\mathbf{k},-\omega)P_{ip}(\mathbf{k})|k^{-y}| \\ &-4P_{im}(\mathbf{k}\epsilon_{mnp}\Omega_n\hat{G}_0(\mathbf{k},\omega)P_{jp}(\mathbf{k})D_0|k|^{-y} \\ &+8D_0P_{im}P_{js}\epsilon_{mnp}\epsilon_{srq}\Omega_n\Omega_r|\hat{G}_0(\mathbf{k},\omega)|^2P_{pq}(\mathbf{k})|k|^{-y}.\end{aligned} \tag{41}$$

This *effective model*, although not exact, retains two basic effects of rotation, namely, anisotropicity and breakdown of parity invariance. We now use this effective noise with a variance (41) together with the *effective model* in the incompressible limit (40) to calculate the inertial range of the two-point equal time correlations of different velocity components. The calculational advantage of the effective model Eq. (38) stems from the fact that its bare propagator is identical to $\hat{G}_0$, i.e., isotropic. Below we use Eq. (40) as a starting point for our analyses below.

Formally, fluctuation corrections to the bare propagator is given by the Dyson equation

$$\hat{G}^{-1}_{\alpha\beta}(\mathbf{k},\omega) = \hat{G}_0^{-1}\delta_{\alpha\beta}(\mathbf{k},\omega) - \Sigma_{\alpha\beta}(\mathbf{k},\omega), \tag{42}$$

where $\hat{G}_{\alpha\beta}(\mathbf{k},\omega)$ is the *renormalized* (fluctuation corrected) propagator, $\hat{G}_0(\mathbf{k},\omega)$ is the bare propagator and $\Sigma_{\alpha\beta}(\mathbf{k},\omega)$ is self-energy which arises due to the non-linear terms and contain fluctuation corrections. The self-energy at the one-loop level is given by (in terms of the bare propagator $\hat{G}_0(\mathbf{k},\omega)$)

$$\begin{aligned}
\Sigma_{ls} &= -\lambda^2 P_{lmn}(\mathbf{k}) \int \frac{d\omega_1}{2\pi}\frac{d^dq}{(2\pi)^d}\frac{1}{\omega_1^2+\nu_0 q^4}\frac{1}{-i\omega_1+\nu_0(\mathbf{k}-\mathbf{q})^2} \\
&\quad \times P_{nps}(\mathbf{k}-\mathbf{q})\langle\phi_m(\mathbf{q},\omega_1)\phi_p(-\mathbf{q},-\omega_1)\rangle \\
&= -\frac{\lambda^2}{2}P_{lmn}(\mathbf{k})\int \frac{d\omega_1}{2\pi}\frac{d^dq}{(2\pi)^d}\frac{P_{nps}(\mathbf{k}-\mathbf{q})}{\omega_1^2+\nu_0^2 k^4} \\
&\quad \frac{2D_0|q|^{-y}}{-i\omega_1+\nu_0(\mathbf{k}-\mathbf{q})^2}[P_{mp}(\mathbf{q}) + 2P_{p\alpha}(\mathbf{q})P_{m\beta}\Omega\epsilon_{\alpha z\beta} \\
&\quad \frac{2i\omega_1}{\omega_1^2+\nu_0 q^4} + 4P_{m\alpha}(\mathbf{q})P_{p\beta}\epsilon_{\alpha z\nu}\epsilon_{\beta z\delta}\Omega^2\frac{P_{\nu\delta}}{\omega_1^2+\nu_0 q^4}].
\end{aligned} \tag{43}$$

Expression (43) shows that the self-energy Σ_{ls} has an Ω-independent part which is identical to the one-loop self-energy expression for the non-rotating case, an $O(\Omega)$ part and an $O(\Omega^2)$ part: The $O(\Omega)$ contributions to different elements of Σ_{ls} are (for details see Sec. IX)

$$\Sigma ij(\mathbf{k},\omega=0) \sim P_{il}\epsilon_{jzl}\frac{Dk^2\Omega}{\nu_0^3}\int dq q^{d-1}q^{-y-6}, \tag{44}$$

where Cartesian indices i, j refer to x, y or z, such that the first order solution to the velocity field is given by $\Sigma_{il}\epsilon_{jzl}\phi_j$. It is clear that there are no $O(\Omega)$ corrections to Σ_{zz}. Furthermore, the zero-frequency one-loop contribution to Σ_{ij} is real. The $O(\Omega^0)$ contribution to Σ_{zz} has infra-red divergence. Demanding self-consistency and noting that there are no fluctuation corrections to the noise strength D_0 [18], we find that the scale-dependent renormalized viscosity $\nu(k) \sim \nu k^{-4/3}$, where we have ignored effects of anisotropy. Noting that in the

long wavelength limit, the *effective* or *renormalized* viscosity diverges as $\nu k^{-4/3}$ [18], where ν is a (dimensional) numerical coefficient, and using it in place of bare viscosity ν_0 in (44) we obtain

$$\Sigma_{ij}(k) \sim P_{il}\epsilon_{jzl}k^{2/3}M^{-1}, \tag{45}$$

as the $O(\Omega)$ correction to the self-energy. Here, $M^{-1} = 2\Omega/(\nu k^{2/3})$ is the *renormalized* or *scale-dependent* inverse Ekman number. Thus the Ω-dependent corrections will appear as a series in $M^{-1}(k)$. With this scale-dependent self-energy $\Sigma_{ij}(k)$ we now calculate the different elements of the two-point velocity correlation function matrix. While doing this we use the effective equation (40) together with the noise variance (41) which now depends upon the scale-dependent propagator G_{ij}. As for the effective noise, since we have

$$\phi_i(\mathbf{k},\omega) \;=\; f_i(\mathbf{k},\omega) - 2P_{im}\epsilon_{mzp}G_{ps}f_s \tag{46}$$

with $i = x, y, z$, the Ω-dependent part of the effective noise ϕ_j has no contribution from $G_{zz}(\mathbf{k},\omega)$. Explicitly calculating we find

$$\begin{aligned}
G_{11} &= \frac{-i\omega + \nu k^{2/3} - 2P_{xy}\Omega}{(-i\omega+\nu k^{2/3})^2 + 4\frac{k_z^2}{k^2}\Omega^2},\\
G_{12} &= \frac{2P_{yy}\Omega}{(-i\omega+\nu k^{2/3})^2 + 4\frac{k_z^2}{k^2}\Omega^2},\\
G_{13} &= \frac{4P_{yy}P_{zx}\Omega^2 + 2P_{zy}(-i\omega+\nu k^{2/3} - 2P_{xy}\Omega)\Omega}{[(-i\omega+\nu k^{2/3})^2 + 4\frac{k_z^2}{k^2}\Omega^2](-i\omega+\nu k^{2/3})},\\
G_{21} &= \frac{2P_{xx}\Omega}{(-i\omega+\nu k^{2/3})^2 + 4\frac{k_z^2}{k^2}\Omega^2},\\
G_{22} &= \frac{-i\omega + \nu k^{2/3} + 2P_{xy}\Omega}{(-i\omega+\nu k^{2/3})^2 + 4\frac{k_z^2}{k^2}\Omega^2},\\
G_{23} &= \frac{4P_{xx}P_{zy}\Omega^2 - 2P_{zx}\Omega(-i\omega+\nu k^{2/3} - 2P_{xy}\Omega)}{[(-i\omega+\nu k^{2/3})^2 + 4\frac{k_z^2}{k^2}\Omega^2]},\\
G_{33} &= \frac{1}{-i\omega+\nu k^{2/3}},\\
G_{31} &= 0 = G_{32}.
\end{aligned} \tag{47}$$

Let us now consider various correlation functions. We find

$$C_{zz}(\mathbf{k},\omega) \;=\; \langle |v_z(\mathbf{k},\omega)|^2\rangle = \frac{\langle \phi_z(\mathbf{k},\omega)\phi_z(-\mathbf{k},-\omega)\rangle}{\omega^2+\nu^2k^{4/3}}. \tag{48}$$

Thus, the equal time correlator $C_{zz}(k, t = 0) \sim k^{-11/3}$ in the long wavelength limit. In contrast, all other correlators scale differently in the inertial range. Let us consider one of

them, namely, $C_{xx} = \langle |v_x(\mathbf{k},\omega)|^2 \rangle$. We begin by noting that

$$v_x = G_{11}\phi_x + G_{12}\phi_y + G_{13}\phi_z. \tag{49}$$

Correlation C_{xx} may now be calculated from (49) by using the effective noise variance. A full calculation keeping the proper anisotropicity is algebraically difficult; hence we perform a scaling level calculation ignoring anisotropic amplitudes. Keeping in mind that the propagators that appear in the noise variance should be renormalized propagators, we find that, since ϕ_i does not receive any contribution from G_{zz}, we find

$$C_{xx}(\mathbf{k},\omega) = \langle |G_{11}\phi_x + G_{12}\phi_y + G_{13}\phi_z|^2 \rangle. \tag{50}$$

Noting that the effective noise ϕ_i involves all elements of the propagator matrix except for G_{zz}, the leading scale dependence of C_{xx} is given by $C_{xx} \sim k^{-3}$ to the leading order in Ω^2 (again we have ignored the anisotropy). In a similar way, C_{yy} also scales as k^{-3}. We have already shown above that C_{zz} scales as $k^{-11/3}$. Thus C_{xx} and C_{yy} scales very differently from C_{zz}. The latter scales same as in $3d$, where as C_{xx} and $C_{yy} \sim k^{-3}$ is less steep than C_{zz}. For each of these we however only extract the scale-dependence and no result on the dependences on k_z and k separately. Let us see what our results above mean for one-dimensional energy spectra. Since $\langle |v_z(\mathbf{k},t)^2\rangle$ scales differently from $\langle |v_j(\mathbf{k},t)|^2\rangle$, $j = x, y$, it is natural to define two one-dimensional energy spectra: $E_{\|}(k) \sim k^2 \langle |v_z(\mathbf{k},t)^2\rangle \sim k^{-5/3}$ and $E_{\perp} \sim k^2 \langle |v_j(\mathbf{k},t)|^2\rangle \sim k^{-1}$, $j = x, y$, where the factor k^2 appears due to the isotropic phase factor in $3d$ (since we ignored the anisotropic coefficient we have used the $3d$ isotropic phase factor). As our scaling level results do not distinguish between different directions in the Fourier space, the energy spectra obtained above also do not have any anisotropy. Finally, in a related issue, our results here provide for a naturally occurring example of a situation where scaling are different in different directions, see, e.g., Ref. [30] and references therein for examples of models exhibiting such behavior. Let us now compare with the existing results: Recent direct numerical studies [13] of the forced Navier-Stokes equation (4) for a rotating fluid for large rotation yield for one-dimensional spectrum $E(\mathbf{k}_\perp) \sim k_\perp^{-2}$, where $\mathbf{k}_\perp$ is perpendicular to $\mathbf{\Omega}$. Similarly numerical solutions of shell model equations [12] for rotating turbulence yield for energy spectrum $E(k) \sim k^{-2}$ for large rotation (shell models, being one-dimensional models, do not distinguish between different directions). Weak turbulence theory [14] approach as well suggests a spectrum $\sim k_\perp^{-2}$. In contrast, our one-loop mode-coupling calculations for *small*-Ω yields spectra k^{-1}, less steep than those obtained in earlier

numerical approaches. However, two things should be kept in mind while comparing our results with results obtained from other approaches: (i) our perturbative calculation and the effective equation are valid only for small-Ω, (ii) we did not distinguish between different directions (due to intractable and complicated nature of the underlying one-loop integrals), and our consequent usage of the $3d$ phase factor is actually not appropriate. An anisotropic phase factor is likely to change the scaling of energy spectra here. These make direct comparisons with other existing results difficult.

V. CONCLUSION

In this article we have analyzed the effects of rotation on the scaling properties of $3d$ homogeneous incompressible turbulence. We have used a two-pronged strategy: First of all, we set up hierarchical relations between structure functions of different orders by using an approach used in Refs. [24, 25]. Unlike the isotropic case, there are no closed relations involving the third-order structure functions. Moreover, second-order structure functions appear in those relations. Thus simple analogs of the well-known von Karman-Howarth relation for isotropic fluid turbulence do not exist here. Furthermore, mixed second-order structure functions made of pressure gradients and differences of velocity components appear in these relations. We are able obtain a differential form analog of the von Karman-Howarth relation of non-rotating fluid turbulence. However, this is non-integrable. All these features are in contrast to the results for isotropic turbulence ($\mathbf{\Omega} = 0$). In the limit of large-$\mathbf{\Omega}$ these relations yield exact relations between certain second-order velocity structure functions and mixed second order structure functions of pressure gradient and differences of velocity components. The overall structures of the hierarchical relations suggest that the scaling properties of velocity structure functions involving one or more in-plane (plane perpendicular to the axis of rotation) is expected to be very different from $S_{2,0,0} = \langle(\Delta v_z)^2\rangle$. In a similar way, we are able to derive exact relations between third-order velocity structure functions and mixed third-order structure functions involving two factors of velocity differences and one factor of pressure gradient difference in the limit of large rotation. Again, similar to the case of the second-order structure functions third-order structure function $S_{3,0,0} \equiv \langle(\Delta v_z)^3\rangle$ does not appear in any of these relations, reinforcing further the effects of anisotropy due to rotation. We then ask: Are the turbulent flows at $\Omega \to \infty$ statistically same as pure $2d$

turbulence? As we have discussed above, the two are *not* identical, a fact which is clearly brought out by the respective structure function hierarchies in the two cases. Thus our conclusions are in agreement with that of Ref. [14]. With these exact relations at hand, we then embark upon an explicit calculation of the scaling of different two-point equal time velocity correlation functions. In order to simplify the ensuing calculations we use an effective model and use a one-loop approximation for our purposes. The resulting one-loop integrals are complicated due to the anisotropic nature of the system. Treating them at the scaling level, i.e., ignoring anisotropy, we obtain $\langle |v_z(\mathbf{k},t)|^2\rangle \sim k^{-11/3}$, identical to the result in three dimensions. In contrast all other two-point equal time velocity correlators display a scaling $\sim k^{-3}$. The latter result imply a three-dimensional spectra (again at the scaling level) $\sim k^{-1}$, different from the conclusions arrived at by using other methods for large rotation. We would like to emphasize that our mode-coupling results are only at the scaling level and hold for small rotation; hence these are only illustrative of the anisotropic scaling due to rotation. In order to compare with numerical results or results obtained by other analytical means, more elaborate calculations, keeping the anisotropic coefficients of the one-loop integrals, should be performed. Our results, especially on the relations between different structure functions, may be tested by direct numerical simulations or experiments: One needs to calculate/measure, e.g., the two sides of the relations (26) or (29, and determine their validity.

Our hierarchical relations are *exact*; however they are not closed and cannot be solved. Despite that they bring out two important results (albeit indirectly): (i) scaling of the two-point structure function $S_{2,0,0}$ is likely to be different from the scaling of all other two-point structure function, (ii) statistical properties of rotating turbulent flows are not the same as those of pure $2d$ flows. In contrast, our perturbative calculations are approximate, but they still provide explicit results on the scaling properties of the various two-point velocity correlation functions which are not in contradiction with the exact hierarchical relations. One-loop diagrammatic calculations presented here suffer from several limitations which are well-documented in standard literature. Despite these difficulties we are able to obtain useful results from them and open up many new relevant and interesting questions for future studies. As a next step, it would be useful to account for the anisotropy in the one-loop integrals and find out the anisotropic scaling functions in the expressions of various correlators. Further, it would be interesting to find out perturbatively or numerically whether

large rotation may lead to unequal dynamic exponents for different correlation functions. If that happens then it would be a natural example of what is known as *weak dynamical scaling* in the literature. Until now the latter has been observed only in simple model studies [31]. Finally some shell model studies [12] indicated that as Ω increases intermittency corrections to the Kolmogorov's simple scaling exponents decrease and finally disappear for high Ω. Given the fact that shell models do not distinguish between various directions, it would be useful to address this question by using Direct Numerical Simulations (DNS) of the $3d$ Navier Stokes equation in a rotating frame and see whether multiscaling disappears only for the structure functions made of in-plane velocity components and survives for the direction parallel to $\mathbf{\Omega}$. In general our results suggest that experiments or DNS studies should measure the scaling of structure functions $\langle(\Delta v_j)^n\rangle$ for $j = z, r, \theta$ with positive n. It should be tested whether $\langle(\Delta v_z)^n\rangle$ shows simple scaling for large-$\mathbf{\Omega}$ for all $n > 0$. We hope that our results will motivate more detailed experimental work as well in the directions as discussed above.

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VII. APPENDIX I: CALCULATION OF THE DISSIPATIVE ANOMALY

We present explicit evaluation of dissipative anomaly D: We begin with

$$\begin{aligned}
&\langle \nu_0 \Sigma_i \partial_i^2 \exp[\lambda_z \Delta v_z + \eta_1 \Delta v_r + \eta_2 \Delta v_\theta]\rangle \\
= \; & \nu_0 \langle \lambda_z [\nabla_2^2 v_z(\mathbf{x}_2) - \nabla_1^2 v_z(\mathbf{x}_1)] \ddagger_0 \rangle + \nu_0 \langle \eta_1 [\nabla_2^2 v_r(\mathbf{x}_2) - \nabla_1^2 v_r] \mathcal{Z}_0 \rangle + \nu_0 \langle \eta_2 [\nabla_2^2 v_\theta(\mathbf{x}_2) - \nabla_1^2 v_\theta(\mathbf{x}_1)] \mathcal{Z}_0 \rangle \\
+ \; & 2\nu_0 [\langle \lambda_z^2 (\nabla_i v_z)^2 \rangle + \langle \eta_1 (\nabla_i v_r)^2 \rangle + \langle \eta_2 (\nabla_i v \theta)^2 \rangle \\
+ \; & 2\lambda_z \eta_1 \langle (\nabla_i v_z)(\nabla_i v_r) \rangle + 2\eta_1 \eta_2 \langle (\nabla_i v_r)(\nabla_i v_\theta) + 2\eta_2 \lambda_z (\nabla_i v_\theta)(\nabla_i v_z) \rangle] \mathcal{Z}_0 \rangle. \qquad (51)
\end{aligned}$$

The left hand side of Eq. (51), upon expansion yields various structure functions of different order multiplied by a factor of ν_0. Since structure functions themselves are finite in the limit

$\nu_0 \to 0$, the left hand side of (51) vanishes in the limit $\nu_0 \to 0$. Thus we obtain

$$
\begin{aligned}
D &\equiv \nu_0\langle \lambda_z[\nabla_2^2 v_z(\mathbf{x}_2) - \nabla_1^2 v_z(\mathbf{x}_1)]\ddagger_0\rangle + \nu\langle \eta_1[\nabla_2^2 v_r(\mathbf{x}_2) - \nabla_1^2 v_r]\mathcal{Z}_0\rangle + \nu\langle \eta_2[\nabla_2^2 v_\theta(\mathbf{x}_2) - \nabla_1^2 v_\theta(\mathbf{x}_1)]\mathcal{Z}_0\rangle \\
&- 2[\langle \lambda_z^2 \epsilon_z\rangle + \langle \eta_1 \epsilon_r\rangle + \langle \eta_2 \epsilon_\theta\rangle + 2\lambda_z\eta_1\nu(\nabla_i v_z)(\nabla_i v_r) + 2\eta_1\eta_2\nu(\nabla_i v_r)(\nabla_i v_\theta) \\
&+ 2\eta_2\lambda_z\nu(\nabla_i v_\theta)(\nabla_i v_z)]\mathcal{Z}_0\rangle, \qquad (52)
\end{aligned}
$$

with $\epsilon_z = (\nabla_i v_z)^2$, $\epsilon_r = (\nabla_i v_r)^2$, $\epsilon_\theta = (\nabla_i v_\theta)^2$.

VIII. APPENDIX II: EFFECTIVE NOISE VARIANCE

By definition

$$
\begin{aligned}
&\langle \phi_i(\mathbf{k},\omega)\phi_j(-\mathbf{k},-\omega)\rangle = \langle f_i(\mathbf{k},\omega) f_j(-\mathbf{k},-\omega)\rangle \\
&-\langle f_i(\mathbf{k},\omega) 2P_{im}(\mathbf{k})\epsilon_{mnp}\Omega_n G_0(-\mathbf{k},-\omega) f_p(-\mathbf{k},-\omega)\rangle \\
&-\langle 2P_{im}(\mathbf{k})\epsilon_{mnp}\Omega_n G_0(\mathbf{k},\omega) f_p(\mathbf{k},\omega) f_j(-\mathbf{k},-\omega)\rangle \\
&+\langle 2P_{im}\epsilon_{mnp}\Omega_n G_0(\mathbf{k},\omega) f_p(\mathbf{k},\omega) \\
&\times 2P_{js}(\mathbf{k})\epsilon_{srq}\Omega_r G_0(-\mathbf{k},\omega) f_q(-\mathbf{k},-\omega)\rangle. \qquad (53)
\end{aligned}
$$

Substituting from Eq. (6) we obtain Eq. (41).

IX. APPENDIX III

Here we write the full expressions of the Ω-independent I_0 and $0(\Omega)$ part I_Ω:

$$
\begin{aligned}
I_0 &= P_{lmn}(\mathbf{k}) \int \frac{d^3q}{(2\pi)^3} \frac{2D_0 P_{mp}(\mathbf{q}) O_{nps}(\mathbf{q})|q|^{-y}}{2\nu_0^2 q^2[q^2 + (\mathbf{k}-\mathbf{q})^2]}, \qquad (54) \\
I_\Omega &= \frac{\lambda^2}{2} P_{lmn}(\mathbf{k}) \int \frac{d^3q}{(2\pi)^3} 4D_0\Omega |q|^{-y} P_{nps}(\mathbf{k}-\mathbf{q}) P_{p\alpha}(\mathbf{q}) P_{m\beta}(\mathbf{q})\epsilon_{\alpha z\beta} \\
&\times \left[-\frac{3}{2\nu_0^2 q^2[q^2 + (\mathbf{k}-\mathbf{q})^2]}\right]. \qquad (55)
\end{aligned}
$$

The above integrals are formally divergent as the external wavevector $\mathbf{k} \to 0$. Self-consistency is achieved for a scale-dependent viscosity $\nu(k)\nu k^{-4/3}$ and scale-dependent inverse Ekman number $M(k) \sim \nu k^{2/3}/2\Omega$.

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